STABILITY OF THE DOMINANT MODE OF THE NONLINEAR WAVE EQUATION IN A CUBIC MEDIUM

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In a parabolic approximation, a criterion has been found for the stability of the dominant mode of a scalar wave equation for a cubic inertialess medium. The criterion obtained is used to demonstrate the stability of one-dimensional and two-dimensional modes, and the instability of a three-dimensional mode, with respect to small perturbations of amplitude and phase.

One important problem in the theory of self-focusing is the stability of steady-state solutions of a nonlinear wave equation [1-4]. In the present work an investigation was made of the stability of the dominant mode in the one-dimensional, two-dimensional, and three-dimensional cases, with respect to small perturbations of the amplitude and the phase, for a cubic inertialess medium.

The equation describing the propagation of the enveloping electrical field E of a light beam, in the parabolic approximation, has the form [1]

$$i\frac{\partial E}{\partial z} + \Delta_2 E + |E|^2 E = 0 \qquad \left(\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{1}$$

where the wave is propagated along the x axis.

With $\gamma > 0$, Eq. (1) has a denumerable set of solutions of the form $E_n = A_n(r)\exp(i\gamma z)$, where A_n satisfies the equation [5]

$$\frac{d^2A_n}{dr^2} + \frac{1}{r}\frac{dA_n}{dr} - \gamma A_n + A_n^3 = 0 \tag{2}$$

and the boundary conditions

$$\left.\frac{dA_n}{dr}\right|_{r=0} = 0, \quad A_n(\infty) = 0, \quad r = (x^2 + y^2)^{1/2} \quad (n = 0, 1, 2, \ldots)$$

Let us consider the stability of the dominant mode $E_0 = A_n\exp(i\gamma z)$, for which $A_0 > 0$ at all values of r and the operator $L_0 = -\Delta_2 + \gamma - A_0^2$ is not negative, where A_0 is an eigenfunction of L_0 with a null eigenvalue. Since A_0 nowhere reverts to zero, it is a fundamental function of the operator L_0, and the null eigenvalue is the least. Substituting into (1)

$$E = (A_0 + \delta\psi)\exp(i\gamma z) \quad (\delta\Psi = (u + iv)\exp(\Omega z))$$

and leaving only terms of the first order with respect to u and v, we obtain

$$\Omega u = L_0 v, \quad \Omega v = -L_1 u \quad (L_1 = L_0 - 2A_0^2) \tag{3}$$

Eliminating v from the equations, we arrive at the eigenvalues [3]

$$-\Omega^2 u = L_0 L_1 u \tag{4}$$

Since the operator L_0 has a null eigenvalue, in a complete Hilbert space, the finite inverse operator L_0^{-1} does not exist. From (3) it follows that

$$\Omega\langle A_0 | u\rangle = \langle A_0 | L_0 u\rangle = \langle L_0 A_0 | u\rangle = 0 \quad \left(\langle A_0 | u\rangle = \int A_0 u\,dx\,dy\right)$$

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therefore, with $\Omega \neq 0$, all the solutions, u, of system (3) are orthogonal with respect to A_0 (here $\langle A_0 | u \rangle$ is a scalar product). Since solutions of system (3) with $\Omega \neq 0$ are under discussion, it is sufficient to investigate Eq. (4) in the subspace of functions orthogonal to A_0. In this subspace, the inverse operator L_0^{-1} can be applied to (4).

$$-\Omega^2 L_0^{-1} u = L_1 u, \qquad \langle A_0 | u \rangle = 0$$

It follows from the variational principle that the least eigenvalue of $-\Omega_0^2$ is equal to [6]

$$-\Omega_0^2 = \min \frac{\langle \varphi | L_1 | \varphi \rangle}{\langle \varphi | L_0^{-1} | \varphi \rangle} \qquad (\langle \varphi | A_0 \rangle = 0) \tag{5}$$

With $\langle \varphi | A_0 \rangle = 0$, the value of $\langle \varphi | L_0^{-1} | \varphi \rangle$ is positive. Therefore, it is sufficient to investigate the arbitrary minimum of the functional $G = \langle \varphi | L_1 | \varphi \rangle$. If min $G < 0$, there exist exponentially increasing perturbations and E_0 is unstable; if min $G \geq 0$, all values of Ω are purely imaginary or equal to 0, and E_0 is a stable solution. It was shown in [3] that the absolute minimum of G is negative and, on this basis, a conclusion was reached with respect to the instability of E_0. However, a negative character of the absolute minimum of G is a necessary but not sufficient condition for the existence of exponentially growing perturbations. It is shown below that the arbitrary minimum of G is equal to zero and, by the same token, the stability of E_0 with respect to small perturbations of amplitude and phase is demonstrated.

Using the method of indeterminate Lagrangian multipliers, we can obtain an equation for a function of ψ, minimizing the functional G:

$$L_1 \psi = \lambda \psi + \alpha A_0 \tag{6}$$

where λ and α are constants, determined from the conditions for orthogonality and normalizing

$$\langle \psi | A_0 \rangle = 0 \; \langle \psi | \psi \rangle = 1, \; \min G = \min \lambda$$

Expanding ψ and A_0 with respect to a complete orthonormalized system of eigenfunctions $L_1 (L_1 \psi_n = \lambda_n \psi_n)$ and substituting these expansions into (6), we obtain

$$\psi = \alpha \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \lambda} \psi_n, \qquad (c_n = \langle \psi_n | A_0 \rangle)$$

The condition $\langle A_0 | \psi \rangle = 0$ leads to an equation for determining λ:

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n - \lambda} = 0 \tag{7}$$

A computer analysis of the spectrum of the operator L_1 showed that L_1 has one negative eigenvalue $\lambda_1 = -5.44\gamma$, and a second eigenvalue $\lambda_2 = 0$; the corresponding eigenfunctions are orthogonal with respect to A_0. As a proof we differentiate (2) with respect to the parameter γ:

$$L_0 \frac{\partial A_0}{\partial \gamma} + A_0 - 2A_0^2 \frac{\partial A_0}{\partial \gamma} = L_1 \frac{\partial A_0}{\partial \gamma} + A_0 = 0$$

or

$$L_1 \partial A_0 / \partial \gamma = -A_0 \tag{8}$$

Multiplying (8) on the left by the eigenfunctions L_1 corresponding to $\lambda_2 = 0$, we obtain

$$-\langle \psi_2 | A_0 \rangle = \left\langle \psi_2 | L_1 | \frac{\partial A_0}{\partial \gamma} \right\rangle = \left\langle L_1 \psi_2 | \frac{\partial A_0}{\partial \gamma} \right\rangle = 0$$

It follows from this that $c_1 = \langle \psi_2 | A_0 \rangle = 0$. We note that $c_1 \neq 0$, since ψ_1, as a fundamental function of the operator L_1, does not have a null value and cannot be orthogonal with respect to A_0, which also nowhere reverts to zero. Thus, the least value of λ_0 in Eq. (7) lies between λ_1 and $\lambda_3 > 0$. To determine the sign of λ_0 it is sufficient to determine $f(0)$ since, in the interval $\lambda_1 < \lambda < \lambda_3, f(\lambda)$ rises monotonically from $-\infty$ to $+\infty$. With $f(0) > 0$, $\lambda_0 < 0$, with $f(0) \leq 0$, $\lambda_0 \geq 0$.

From (7) and (8) it follows that

$$f(0) = \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n} = \langle A_0 | L_1^{-1} | A_0 \rangle = -\left\langle A_0 | \frac{\partial A_0}{\partial \gamma} \right\rangle = -\frac{1}{2} \frac{dI}{d\gamma}$$

where $I = \langle A_0 | A_0 \rangle$ is the energy of the dominant mode.

The solution of Eq. (2) can be written in the form

$$A_0 = \sqrt{\gamma}\varphi_0(\sqrt{\gamma}r)$$

since

$$I = 2\pi\int_0^\infty A_0^2 r\,dr = 2\pi\int_0^\infty \varphi_0^2(\rho)\,\rho\,d\rho$$

does not depend on γ and $dI/d\gamma = 0$. Consequently, $f(0) = 0$ and the arbitrary minimum of $G = 0$. Thus, all the values of Ω of problem (3) are purely imaginary or equal to zero. A computer analysis of system (3) also showed the absence of solutions with real values of $\Omega \neq 0$. From this the conclusion can be drawn that, in a linear approximation with respect to the perturbation, the dominant mode E_0 is stable.

An analogous train of reasoning can be carried through for the one-dimensional and three-dimensional Eq. (2), and it can be shown that the stability of the dominant mode is also determined by the sign of the derivative of its energy with respect to the parameter γ. In the one-dimensional case the dominant mode has the form

$$A_0 = \sqrt{2\gamma}/\mathrm{ch}(\sqrt{\gamma}x)\ ;$$

therefore, its energy is equal to

$$I = \langle A_0 | A_0 \rangle = \int_0^\infty \frac{2\gamma\,dx}{\mathrm{ch}^2(\sqrt{\gamma}\,x)} = 4\sqrt{\gamma}$$

$$dI/d\gamma = 2/\sqrt{\gamma} > 0$$

Consequently, the arbitrary minimum of $G \geq 0$, and the one-dimensional dominant mode in a cubic medium is stable. In the three-dimensional case, the dominant mode can be represented in the form

$$A_0 = \sqrt{\gamma}\,\psi_0(\sqrt{\gamma}\,r), \quad I = 4\pi\int_0^\infty A_0^2 r^2\,dr = \frac{4\pi}{\sqrt{\gamma}}\int_0^\infty \psi_0^2(\rho)\,\rho^2\,d\rho$$

Since

$$\frac{dI}{d\gamma} = -\frac{2\pi}{\gamma^{3/2}}\int_0^\infty \psi_0^2(\rho)\,\rho^2\,d\rho < 0$$

a dominant mode with spherical symmetry in a cubic inertialess medium is unstable. Computer calculations have shown that, in the case of spherical symmetry, system (3) with $\Omega = 5.9\gamma$ has a solution satisfying the boundary conditions

$$\left.\frac{du}{dr}\right|_{r=0} = \left.\frac{dv}{dr}\right|_{r=0} = 0, \qquad u(\infty) = v(\infty) = 0$$

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